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Optimal Aircraft Routing in General Wind Fields

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Introduction

AIRCRAFT navigation involves routing and guidance of airplanes as a part of flight planning. In this Note, optimal aircraft routing refers to route determination for an airplane traveling horizontally between two given points so that the transit time is minimized, assuming that the meteorological conditions (in particular, the wind field) are fully known beforehand for the complete passage. Aircraft guidance refers to guiding the aircraft along this route.

The determination of the minimal-time path for an airplane was first accomplished by Zermelo [1] in a lecture in Prague, which gave rise to a number of publications in this field, including those in which the analogy between optical and minimum-flight problems was pointed out [2]. The actual interest in the problem came after the Second World War, when regular long-distance flights came into operation. After some less successful attempts, a useful manual method for the calculation of minimal-flight paths was developed by introducing time fronts analogous to wave fronts in geometrical optics [3].

The introduction of the computer made this manual method obsolete. The obvious next step was to program appropriate methods such as applications of the calculus of variations and graph theory for computer application. Although the minimal-flight problem can be formulated very simply as a problem of the calculus of variations, the solution must generally be obtained by iteration so that apart from the fact that convergence problems may occur, the iteration process could converge to a relative minimum instead of converging to an absolute minimum. On the other hand, the graph method always yields an absolute minimum, but at the cost of large computation time and memory space.

As an ultimate application of the calculus of variations, an extension of the technique of neighboring optimal control was introduced to compute near-optimal trajectories in general wind fields, starting from nominal solutions to the Zermelo problem obtained with simple analytical or zero wind fields [4]. In [5], it was claimed that the excellent performance of this approach in practice is achieved because winds typically vary in a smooth manner and do not contain many sharp nonlinearities or discontinuities. However, the wind does not generally vary in a smooth manner. Consequently, the choice of a nominal solution is not always obvious and neither is the convergence to a near-optimal trajectory, as we will see in the next section.

Inefficient routing may result in excess fuel burn and, accordingly, in excess emissions. Therefore, combining the approaches of the calculus of variations and graph theory, we propose a method that always yields an optimal solution with moderate computational effort and memory space. The minimal-flight problem is a simple example of the control problem of Bolza [6] from the calculus of variations. Integration of the model equations and varying the initial heading yields a one-parameter family of solutions (extremals) emanating from the point of departure and, under certain conditions, continuous in their dependence on this parameter. New extremals can be inserted at a specific time if the distance between two adjacent extremals becomes too large at that time. If this distance is chosen to be sufficiently small, the starting values of the new extremal can be obtained by linear interpolation between the corresponding values of its neighbors. In this way, a network is built up containing points that can be reached along minimal-time tracks after a certain number of time steps. The ultimate minimal-time track from origin to destination is obtained by selecting that extremal, which ends closest to the destination. This method was tested extensively during the 1970s at the Royal Netherlands Meteorological Institute in many practical situations related to minimal-time ship routing [7].

In the following section, the problem of Bolza is discussed in relation to minimal-time aircraft routing for constant airspeed. Next, this discussion is generalized for airspeeds, which may depend on position, time, and heading, and the relation with minimal-time ship routing is elucidated. Conclusions are presented in the last section.

Computation of a Least-Time Track

To simplify the computations, the navigation area is mapped conformally onto a plane (for instance, by stereographic projection). Introducing a Cartesian coordinate system with coordinates x_1 and x_2, the equations of motion read

$$\dot{x}_1 = V\cos\theta + u_1(t, x_1, x_2) \tag{1}$$

$$\dot{x}_2 = V\sin\theta + u_2(t, x_1, x_2) \tag{2}$$

where V is the airspeed, which is the speed of the aircraft relative to the ambient air. It is assumed here that the flight altitude and airspeed have been chosen to minimize the costs of flight, which include the costs of the consumed fuel and other costs proportional to the flight time [8]. The heading θ is the control variable, and $u_1(t, x_1, x_2)$ and $u_2(t, x_1, x_2)$ denote the x_1 component and the x_2 component of the wind velocity. The airspeed is set to be constant. In this Note, the airspeed may depend on position and time and the heading, reflecting, for instance, the possible fuel-consumption effects of head and tail winds and air temperature.

In the next section, a discussion of the general problem related to ship routing is presented, which includes the present problem as a special case. We now seek to find a continuous control function $\theta(t)$ and a corresponding trajectory $x(t) = (x_1(t), x_2(t))$ satisfying the equations of motion (1) and (2) with initial and end conditions $x_i(0) = x_{i0}$ and $x_i(t_1) = x_{i1} (i = 1, 2)$, which minimizes the time t_1 required to travel from the origin or first waypoint (x_{10}, x_{20}) to the destination or second waypoint (x_{11}, x_{21}). This problem is called the

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control problem of Bolza. The problem of Bolza has been studied extensively in the literature [9]. The function $\theta(t)$ satisfying the control problem of Bolza is an optimal control function, and the arc $x(t)$ is an optimal trajectory. It is assumed here that $x(t)$ is normal, which implies the existence of a one-parameter family of arcs satisfying Eqs. (1) and (2) with the preceding initial and end conditions and containing $x(t)$ as one of its members. Note that the abnormal case is highly singular. Minimization of the flight time contributes to a further reduction of the fuel consumption over the flight. In ship routing, the problem of minimizing the fuel consumption of a ship during its transit over the ocean can be formulated more directly [10]. The necessary condition for the control function $\theta(t)$ and the trajectory $x(t)$ to be optimal is that there exist continuously differentiable multipliers

$$\lambda(t) = (\lambda_0(t), \lambda_1(t), \lambda_2(t)), \lambda_0(t) = \text{constant} \leq 0$$

and a function

$$H(t, x, \theta, \lambda) = \lambda_0 + \lambda_1(V\cos\theta + u_1(t, x_1, x_2)) + \lambda_2(V\sin\theta + u_2(t, x_1, x_2))$$

so that the following conditions hold on $x(t)$:

1) For the first necessary condition, on $x(t)$, the Euler–Lagrange equations

$$\dot{x}_i = H_{\lambda_i}, \qquad \dot{\lambda}_i = -H_{x_i}, \qquad H_\theta = 0 \quad (i = 1, 2)$$

hold. Variables as subscripts denote partial differentiation.

2) For the necessary condition of Weierstrass, along $x(t)$, the inequality

$$H(t, x(t), \theta, \lambda(t)) \leq H(t, x(t), \theta(t), \lambda(t))$$

must hold for any t, for $0 \leq t \leq t_1$. In addition

$$H(t_1, x(t_1), \theta(t_1), \lambda(t_1)) = 0$$

As a consequence of normality, the equality sign of the multiplier λ_0 is excluded. Solutions of the Euler–Lagrange equations with continuous control functions are called extremals. An optimal trajectory is obviously an extremal. The Euler–Lagrange equations read

$$\dot{\lambda}_1 = -\sum_{i=1}^{2} \lambda_i u_{ix_1} \tag{3}$$

$$\dot{\lambda}_2 = -\sum_{i=1}^{2} \lambda_i u_{ix_2} \tag{4}$$

$$-\lambda_1 \sin\theta + \lambda_2 \cos\theta = 0 \tag{5}$$

Using Eq. (5), we substitute $\sin\theta = \lambda_2/(\lambda_1^2 + \lambda_2^2)^{1/2}$ and $\cos\theta = \lambda_1/(\lambda_1^2 + \lambda_2^2)^{1/2}$ into Eqs. (1) and (2). In view of the nature of the problem, we observe that every part of an optimal trajectory is an optimal trajectory itself, which is a direct consequence of the principle of optimality [11]. This means that every point of an extremal can be considered as an endpoint so that the equation $H = 0$ is supposed to be satisfied for all t $(0 \leq t \leq t_1)$, meaning that the relation

$$\lambda_1(V\cos\theta + u_1) + \lambda_2(V\sin\theta + u_2) = \lambda_0 > 0 \tag{6}$$

must hold along an extremal. Let the initial values of the Lagrange multipliers λ_1 and λ_2 be given by $\lambda_1(0) = \cos a$ and $\lambda_2(0) = \sin a$. Then a one-parameter family of extremals emanating from the point of departure is obtained by integrating Eqs. (1–4) and varying the initial values of the heading $\theta(0) = a$, as shown in Figs. 1 and 2, which are discussed in detail elsewhere [12]. That discussion is summarized here. Figure 1 shows such a one-parameter set of extremals originating from New York on 4 May 1970 at 03:00 Greenwich Mean Time (GMT) with an incremental change of the initial heading of 2 deg. In Fig. 2, a similar set of extremals is shown with Amsterdam as the origin. The Mach number is 0.803. The meteorological information consists of grid-point values for geopotential and temperature in the 300 hPa standard pressure level of 4 May 1970 derived from the northern hemispheric octagonal grid-point system used by the National Meteorological Center, Suitland, Maryland. To account for the dynamic character of the atmospheric circulation, use has been made of the 300 hPa grid

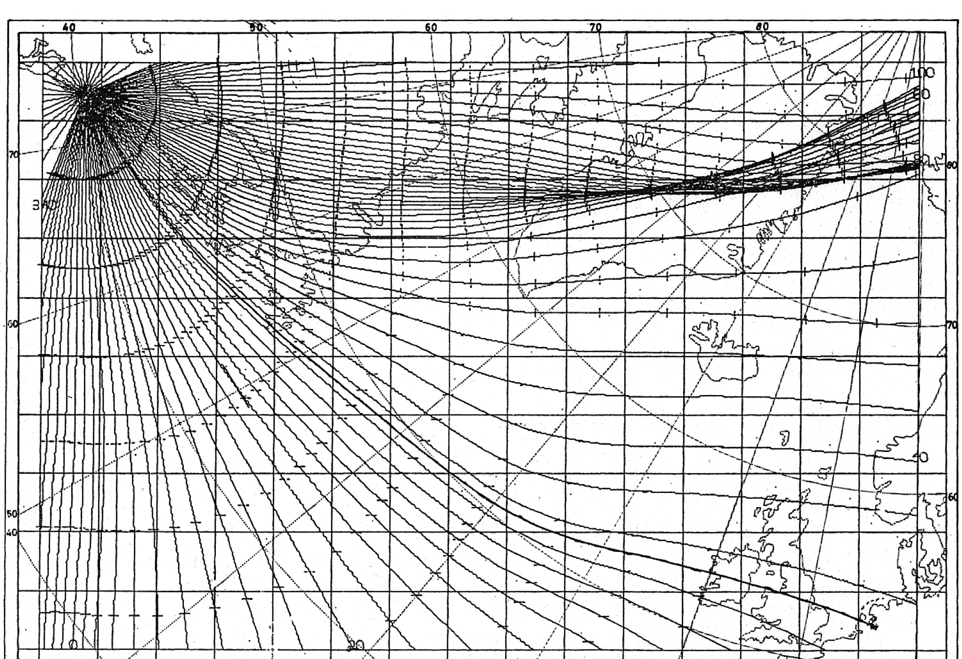

Fig. 1 One-parameter family of extremals with New York as the origin along which the flight time is minimized. Short crossbars indicate the half-hourly transit times. The least-time track to Amsterdam is indicated by the somewhat heavier line.

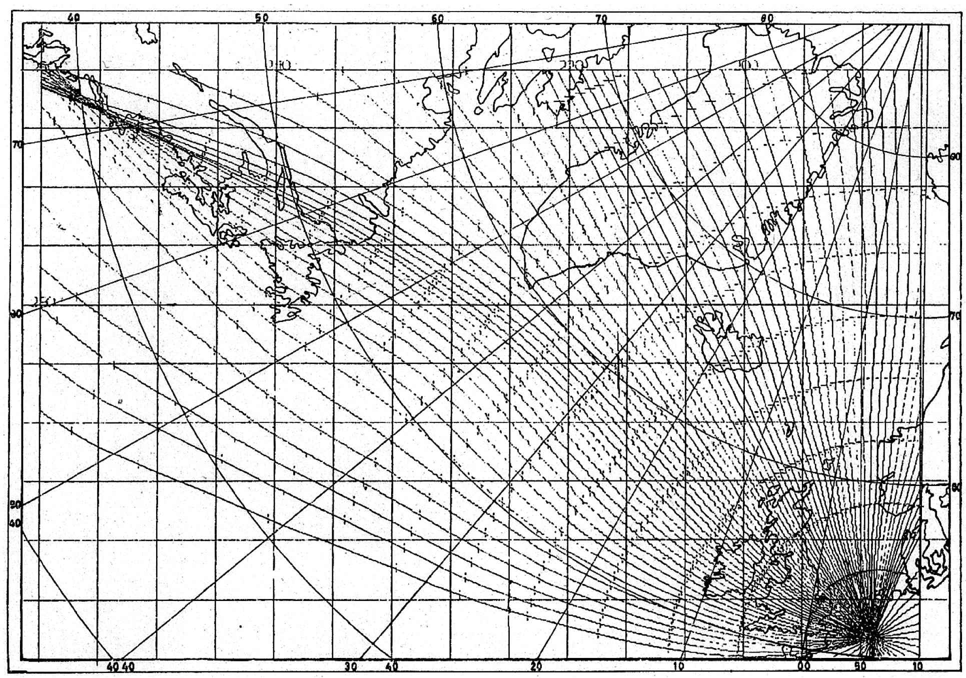

Fig. 2 As in Fig. 1, with Amsterdam as the origin.

representations at the times 00:00 and 12:00 GMT. The wind data are obtained by applying the geostrophic wind equation.

Let us first consider Fig. 1. In spite of the rather regular behavior of the extremals, iterative methods might have convergence problems when optimal routes are computed for which the endpoint is located in regions such as Scotland or Denmark, in which the extremals have grown apart over a considerable distance, due to strong sensitivity to small changes in the initial heading causing instability in the convergence process. Another problem arises if the regularity of the field is disturbed by the focusing effect of extremals over Greenland. When optimal routes are computed with endpoints within the area enclosed by the caustics, two solutions exist, each leading to a minimum with respect to the adjacent trajectories. Convergence will be fast, but the resulting solution may give an absolute or relative minimum, depending on the nominal solution that has been chosen. These observations also apply to Fig. 2, in which New York terminal is located within an area in which the extremals tend to focus. Even in this case, the choice of a nominal solution is not obvious without having any prior knowledge.

In view of the foregoing, we now introduce a computational method that always yields an absolute minimum. Therefore, we are interested in solutions of Eqs. (1–4), which are continuous in their dependence on the parameter a. Application of a theorem on the initial-value problem for systems of ordinary differential equations [13] leads to the following result. Let the right-hand sides of Eqs. (1–4) be continuous and satisfy a Lipschitz condition with respect to x_1, x_2, λ_1, and λ_2. Then $x_i(t, a)$ and $\lambda_i(t, a)$ ($i = 1, 2$ and $0 \leq t \leq t_1$) as solutions of Eqs. (1–4) with $x_i(0) = x_{i0}(i = 1, 2)$ are continuously differentiable with respect to t and continuous in their dependence on the parameter a defined by $\lambda_1(0) = \cos a$ and $\lambda_2(0) = \sin a$.

This result enables us to introduce a numerical method for the solution of optimal control problems in navigation. The method appears to be very suitable in practical cases, because the preceding requirements can be satisfied rather simply. Let us consider the continuous dependence of x_1, x_2, λ_1, and λ_2 on the parameter a from a practical point of view. Regarding the one-parameter family of extremals emanating from the point of departure, we may insert a new extremal between two neighboring extremals at a specific time if the distance between these neighbors becomes too large at that time. If this distance is chosen to be sufficiently small, the starting values of the new extremal can be obtained by linear interpolation between the corresponding values of its neighbors. In this way, a network is built up containing points that can be reached along minimal-time tracks after a certain number of time steps. The ultimate minimal-time track from origin to destination is obtained by selecting that extremal that ends closest to the destination. The essence of the method is demonstrated in Fig. 3.

General Case

In this section, the airspeed may depend on position and time and the heading. The equations of motion read

$$\dot{x}_1 = V(t, x_1, x_2, \theta)\cos\theta + u_1(t, x_1, x_2) \tag{7}$$

$$\dot{x}_2 = V(t, x_1, x_2, \theta)\sin\theta + u_2(t, x_1, x_2) \tag{8}$$

If these equations are related to ship routing, $V(t, x_1, x_2, \theta)$ denotes the maximum speed. The dependence of $V(t, x_1, x_2, \theta)$ on t, x_1, and x_2 reflects the effect of the wave field (wave height and wave direction) on the maximum speed for given values of the heading θ. The functions $u_1(t, x_1, x_2)$ and $u_2(t, x_1, x_2)$ denote the x_1 component and the x_2 component of the ocean current. The Euler–Lagrange equations can be written as

$$\dot{\lambda}_1 = -\sum_{i=1}^{2}\lambda_i(V_{ix_1} + u_{ix_1}) \tag{9}$$

$$\dot{\lambda}_2 = -\sum_{i=1}^{2}\lambda_i(V_{ix_2} + u_{ix_2}) \tag{10}$$

$$\sum_{i=1}^{2}\lambda_i V_{i\theta} = 0 \tag{11}$$

where $V_1 = V\cos\theta$ and $V_2 = V\sin\theta$.

As in the preceding section, the right-hand sides of Eqs. (7–10), in which θ is determined by Eq. (11), must be continuous and satisfy a Lipschitz condition so that x_1, x_2, λ_1, and λ_2 depend continuously on the parameter a defined by $\lambda_1(0) = \cos a$ and $\lambda_2(0) = \sin a$. Application of a theorem on implicit functions [9] to relation (11) shows that the functions $V_{i\theta}$ and $V_{i\theta\theta}(i = 1, 2)$ must be continuous with respect to t, x_1, x_2, and θ and that the condition

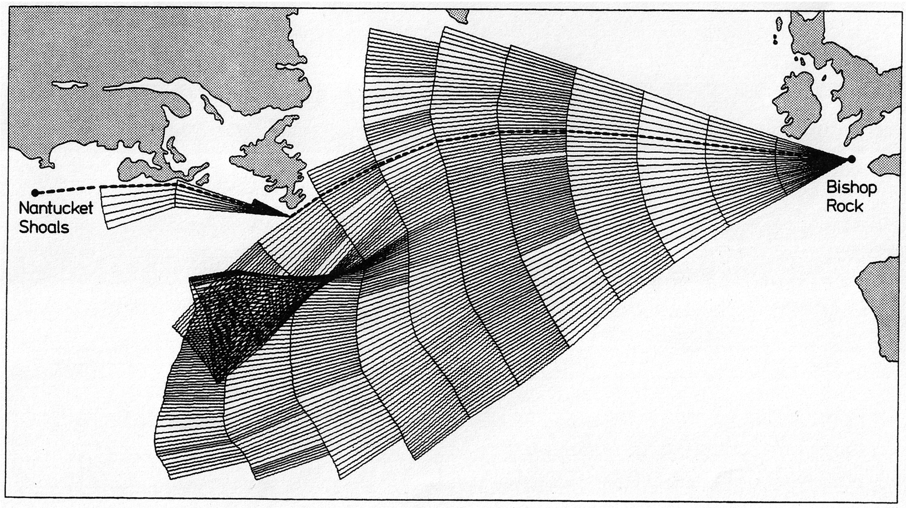

Fig. 3 One-parameter family of extremals along which the sailing time is minimized using wave information over the period 17–23 January 1970, fictitious ship's data and a 12 h time step. The least-time track is indicated by the dashed line.

$$\sum_{i=1}^{2} \lambda_i V_{i\theta\theta} \neq 0 \tag{12}$$

must hold so that θ is a continuous function of t, x_1, x_2, λ_1, and λ_2. Condition (12) implies that the polar velocity diagram giving the ship's velocity as a function of the angle between the ship's heading and the wave direction for fixed values of t, x_1, and x_2 must be convex and that the equality sign for the Legendre condition, being an immediate consequence of the Weierstrass condition, is excluded so that

$$\sum_{i=1}^{2} \lambda_i V_{i\theta\theta} < 0 \tag{13}$$

Conditions (11) and (13) show that the inner product of (λ_1, λ_2) and (V_1, V_2) as a function of θ attains a maximum along an extremal. In addition, relation (6) must hold. An application of the foregoing is illustrated in Fig. 3. For zero wave height, the maximum speed is constant and Eqs. (7–11) are reduced to Eqs. (1–5).

Conclusions

This Note shows that iterative methods applied to the computation of minimal-flight paths on an operational basis are vulnerable, because apart from the fact that convergence problems might occur, the iteration process could converge to a relative minimum instead of converging to an absolute minimum, depending on the nominal solution (first approximation). Because of the importance of aircraft routing for economic and environmental reasons, we therefore propose a method that always yields an absolute minimum, requiring moderate computational effort and memory space. In addition, this method is also capable of handling more general situations.

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